Extended Self-Dual Configurations as Stable Exact Solutions in Born-Infeld Theory

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Abstract

A class of exact solutions to the Born-Infeld field equations, over manifolds of any even dimension, is constructed. They are an extension of the self-dual configurations. They are local minima of the action for riemannian base manifolds and local minima of the Hamiltonian for pseudo-riemannian ones. A general explicit expression for the Born-Infeld determinant is obtained, for any dimension of space-time.

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1   Introduction

The Born-Infeld action [1, 2, 3, 4, 5] has been proposed as the effective theory of open strings with Dirichlet boundary conditions [6, 7, 8, 9, 10]. The D-brane action arising in this way is

$$S = \int d^{p+1}\xi e^{-\phi}\sqrt{-\det\left[G_{\alpha\beta} + B_{\alpha\beta} + bF_{\alpha\beta}\right]} \tag{1}$$

where G, B and ϕ are the pullbacks of the 10 dimensional metric, antisymmetric tensor and dilaton to the D-brane world-volume while F is the curvature of the world volume $U(1)$ gauge field A_α. In the full supersymmetric string theory this action must be extended to a supersymmetric Born-Infeld (B-I) action. If a number of assumptions are made the action (1) becomes the $U(1)$ Maxwell theory in $p+1$ dimensions which arises from dimensional reduction of the $U(1)$ Maxwell theory in 10 dimensions with $N = 1$ supersymmetry.

In order to study the highly nonlinear behavior of the theory (1) with respect to $F_{\alpha\beta}$, we will consider

$$
\begin{aligned}
\phi &= 0 \\
B_{\alpha\beta} &= 0
\end{aligned}
\tag{2}
$$

and $G_{\alpha\beta}$ to be an external metric.

We first obtain a general explicit expression of the Born-Infeld action valid for any dimension of the space-time.

We will then introduce a class of exact solutions to the field equations of the B-I action. This class of configurations may be considered an extension of the self-dual ones. We consider the B-I action over a compact, riemannian manifold M of even dimension

$D = 2n$. We take all the products, with m factors

$$P_m = F \wedge \ldots \wedge F, \tag{3}$$

where F is the curvature of a connection 1-form on a $U(1)$ principle bundle constructed over M. We then define the extended self-dual configurations as the connections 1-forms for which the set $\{P_m \, , \, m = 0, 1, \ldots, n\}$ is mapped into itself by the Hodge dual operation. That is, they satisfy

$$^*P_m \approx P_{n-m}, \tag{4}$$

for $m = 0, \ldots, n$. Of course, the conditions in (4) are not all independent.

It is straightforward to see that the extended self-dual configurations are solutions of Maxwell equations

$$d^*F = 0. \tag{5}$$

In fact,

$$^*F = {}^*P_1 = kP_{n-1} \tag{6}$$

and

$$dP_{n-1} = 0. \tag{7}$$

We will show in this work that the extended self dual configurations are also exact solutions of the Born-Infeld field equations, moreover we will show they are strict minima of the action. When the base manifold is pseudo-riemannian of the form $M \times \mathbb{R}$, M being compact riemannian, we will show that the extended self-dual configurations over M are the minima of the corresponding Hamiltonian. As in the case of instanton solutions to the self-dual equations, given a manifold with a metric over it the solutions may or may not exist. We give in section 3 examples where the extended self-dual

configurations do exist. The canonical connection 1-forms, introduced by Trautmant in [11], over the Hopf fibring

$$S_{2n+1} \longrightarrow \mathbb{C}P_n$$

are in particular examples of extended self-dual configurations. It was also proven there that these connections are solutions to the Maxwell equations.

2 General formula for the determinant

In this section we obtain the general formula for the determinant of $g_{ab} + F_{ab}$ which we will extensively use in this work.

We may reexpress $g + F$, using the properties

$$
\begin{aligned}
g^T &= g, \\
F^T &= -F,
\end{aligned}
\tag{8}
$$

in the following way

$$g + F = M^T D M + F, \tag{9}$$

where D is a diagonal matrix, while M is an orthogonal one.

We then have

$$\det(g + F) = \det\left(D + MFM^T\right) \tag{10}$$

where MFM^T is again an antisymmetric matrix. We denote the elements of D as $\lambda_a \delta_{ab}$ and

$$G \equiv MFM^T.$$

For a general $N \times N$ matrix, we have

$$\det(D + G) = \sum_m \frac{1}{(2m)!\,(N - 2m)!} \epsilon^{a_1 \cdots a_m c_1 \cdots c_m e_{2m+1} \cdots e_N} \epsilon^{b_1 \cdots b_m d_1 \cdots d_m e_{2m+1} \cdots e_N}$$

4

$$\times \lambda_{e_{2m+1}} \cdots \lambda_N G_{a_1 b_1} \cdots G_{a_m b_m} G_{c_1 d_1} \cdots G_{c_m d_m} \tag{11}$$

For a given element $\hat{E}$ of the set $E \equiv \{ \, (e_{2m+1}, \ldots, e_N) :$ each index may take values from 1 to $N \, \}$, the terms in (11)

$$\frac{1}{(2m)!} \epsilon^{a_1 \cdots a_m c_1 \cdots c_m \hat{E}} \epsilon^{b_1 \cdots b_m d_1 \cdots d_m \hat{E}} G_{a_1 b_1} \cdots G_{a_m b_m} G_{c_1 d_1} \cdots G_{c_m d_m} \tag{12}$$

represent the determinant of a $2m \times 2m$ antisymmetric matrix. The determinant of an antisymmetric $N \times N$ matrix is zero if N is odd and for $N = 2m$ one has the following expression, called the *pfaffian*,

$$\frac{1}{2^{2m} \, (m!)^2} \left(G_{a_1 b_1} \cdots G_{a_m b_m} \epsilon^{a_1 \cdots a_m b_1 \cdots b_m \hat{E}} \right)^2 . \tag{13}$$

We then have for (11) the expression

$$\det (D + G) \;=\; \sum_m \frac{1}{(N - 2m)! 2^{2m} \, (m!)^2} \left(G_{a_1 b_1} \cdots G_{a_m b_m} \epsilon^{a_1 b_1 \cdots a_m b_m e_{2m+1} \cdots e_N} \right)^2$$
$$\times \lambda_{e_{2m+1}} \cdots \lambda_{e_N} . \tag{14}$$

We may now (14) in terms of the original g and F matrices, using that $\det M = 1$, we obtain

$$\det (g + F) \;=\; \sum_m \frac{1}{(N - 2m)! 2^{2m} \, (m!)^2} \left(F_{a_1 b_1} \cdots F_{a_m b_m} \epsilon^{a_1 b_1 \cdots a_m b_m e_{2m+1} \cdots e_N} \right)$$
$$\times \left(F_{c_1 d_1} \cdots F_{c_m d_m} \epsilon^{c_1 d_1 \cdots c_m d_m l_{2m+1} \cdots l_N} \right) g_{e_{2m+1} l_{2m+1}} \cdots g_{e_N l_N} . \tag{15}$$

We now introduce

$$F \;\equiv\; F_{ab} dx^a \wedge dx^b$$
$$P_m \;\equiv\; \underbrace{F \wedge \ldots \wedge F}_{m} = F_{a_1 b_1} \cdots F_{a_m b_m} dx^{a_1} \wedge dx^{b_1} \wedge \ldots \wedge dx^{b_m}$$
$${}^* P_m \;=\; \sqrt{|g|} \epsilon_{a_1 b_1 \cdots a_m b_m e_{2m+1} \cdots e_N} F^{a_1 b_1} \cdots F^{a_m b_m} dx^{e_{2m+1}} \wedge \cdots \wedge dx^N , \tag{16}$$

5

where $g \equiv \det g_{ab}$.

We then have, the final expression

$$\det (g + F) = |g| \sum_{m=0}^{n} \frac{1}{(N - 2m)! 2^{2m} (m!)^2} {}^* (P_m \wedge {}^*P_m) \tag{17}$$

where $n = \frac{[N]}{2}$.

3 A class of solutions for Born-Infeld field equations

The Born-Infeld theory formulated over a Riemannian manifold M may be described by the following D dimensional action

$$S(A) = \int_M \left(\sqrt{\det (g_{ab} + bF_{ab})} - \sqrt{g} \right) d^D x \tag{18}$$

where g_{ab} is an external euclidean metric over the compact closed manifold M. F_{ab} are the components of the curvature of connection 1-form A over a $U(1)$ principle bundle on M.

We may express the $\det (g_{ab} + bF_{ab})$, using the general formula obtained in (17), as

$$\begin{aligned} \det (g_{ab} + bF_{ab}) &= g \sum_{m=0}^{n} a_m b^{2m *} [P_m \wedge {}^*P_m] \\ &\equiv gW. \end{aligned} \tag{19}$$

The first variation of (18) is given by

$$\delta S(A) = \int_M W^{-\frac{1}{2}} \sum_m m a_m b^{2m} d\delta A \wedge P_{m-1} \wedge {}^*P_m \tag{20}$$

which yields the following field equations

$$\sum_m m a_m b^{2m} P_{m-1} \wedge d \left(W^{-\frac{1}{2}*} P_m \right) = 0. \tag{21}$$

6

We introduce now a set $\mathcal{A}$ of $U(1)$ connection 1-forms over M. They are defined by the following conditions,

$$^*P_m\,(A) = k_m P_{n-m}\,(A) \qquad , m = 0, \ldots, n, \tag{22}$$

where $n = \frac{D}{2}$, that is we assume the dimension D of M to be an even natural number. (22) may be interpreted as an extension of the self duality condition. It is the condition that the Hodge dual transformation maps the set $\{P_m\,, \quad m = 0, \ldots, n\}$ into itself.

We observe that these connections, if they exits in a $U(1)$ principle bundle over M, are solutions of the field equations (21).

In fact, (22) implies

$$^*\,[P_m \wedge {}^*P_m] = k_m{}^*\,[P_m \wedge P_{n-m}] = k_m{}^*P_n \tag{23}$$

but from (22), for $m = n$, we obtain

$$^*P_n = k_n \tag{24}$$

which is constant. We thus have, for these connections,

$$W = \text{constant.} \tag{25}$$

Finally, it results

$$d\left(W^{-\frac{1}{2}*}P_m\right) = k_m W^{-\frac{1}{2}}d\,(P_{n-m}) = 0, \tag{26}$$

showing that (22), if they exits, define a set of solutions to the Born-Infeld field equations.

Let us analyze a particular case of (22). Let us consider $n = \frac{D}{2} = 1$. We then have

$$^*P_1 = {}^*F = k_1. \tag{27}$$

This solution represents a monopole connection over the $D = 2$ manifold M. When M is the sphere S_2, (27) defines the $U(1)$ connection describing the Dirac monopole on the Hopf fibring $S_3 \rightarrow S_2$. The constant k_1 is determined from the condition

$$\int_M F = 2\pi \times \text{integer} \tag{28}$$

which is a necessary condition to be satisfied for a $U(1)$ connection, F being its curvature.

This solution was extended to $U(1)$ connections over Riemann surfaces of any genus in [12, 13, 14] where it was shown that they describe the minima of the hamiltonian of the double compactified $D = 11$ supermembrane dual.

In general, if we consider M to be $\mathbb{C}P_n$ and the $U(1)$ Hopf fibring

$$S_{2n+1} \rightarrow \mathbb{C}P_n, \tag{29}$$

there is a canonical way to obtain g_{ab} and the connection 1-form A satisfying (22) from the metric over S_{2n+1} [11]. If we consider local complex coordinates over S_{2n+1}, z_α $(\alpha = 0, 1, \ldots, n)$, satisfying

$$\bar{z}_\alpha z_\alpha = 1, \tag{30}$$

its line element

$$d\bar{z}_\alpha dz_\alpha$$

may be decomposed into the line element of the base manifold $\mathbb{C}P_n$ and the canonical 1-form connection ω, in the following way

$$d\bar{z}_\alpha dz_\alpha = ds^2 - \omega^2. \tag{31}$$

In the coordinate system introduced in [11] ω has the expression

$$\omega = u^{-1}du + \frac{1}{2}\rho^2 \left(\bar{\zeta}_a d\zeta_a - \zeta_a d\bar{\zeta}_a \right), \qquad a = 1, \ldots, n$$

8

while the kählerian metric is given by

$$ds^2 = h_{ab}d\bar{\zeta}_a d\zeta_b$$

and the curvature

$$\Omega = \mathrm{i}d\omega = \mathrm{i}h_{ab}d\bar{\zeta}_a \wedge d\zeta_b \tag{32}$$

with

$$h_{ab} = \rho^4 \left[\delta_{ab}\rho^{-2} - \zeta_a\bar{\zeta}_b \right]$$

where

$$\rho^2 = \frac{1}{1 + \bar{\zeta}_a\zeta_a}.$$

u is the coordinate in the fiber over $\mathbb{C}P_n$ and ζ_a the local coordinates over $\mathbb{C}P_n$.

It can be show that the Kähler 2-form (32) with the kählerian metric satisfies the extended self-dual conditions (22) [15].

4 Minima of the Born-Infeld action

We show in this section some properties of the set $\mathcal{A}$ of connection 1-forms we have introduced in the previous section. On a given $U(1)$ principle bundle over M there is at most one $\hat{A} \in \mathcal{A}$ modulo flat connections on a trivial bundle. In fact, we may consider the functional

$$w(A) \equiv \int_M \sum_{m=1}^{n} a_m P_m \wedge {}^*P_m. \tag{33}$$

We then have for $\hat{A} \in \mathcal{A}$ and any A on the same $U(1)$ principle bundle

$$w(A) = w\left(\hat{A}\right) + \int_M \sum_{m=1}^{n} a_m \left[P_m(A) - P_m\left(\hat{A}\right) \right] \wedge {}^* \left[P_m(A) - P_m\left(\hat{A}\right) \right] \tag{34}$$

9

since

$$\int_M \sum_{m=1}^n a_m P_m(A) \wedge {}^*P_m\left(\hat{A}\right) = \int_M \sum_{m=1}^n a_m P_m\left(\hat{A}\right) \wedge {}^*P_m\left(\hat{A}\right) \tag{35}$$

(34) implies

$$w(A) \geq w\left(\hat{A}\right), \tag{36}$$

and

$$w(A) = w\left(\hat{A}\right) \tag{37}$$

if and only if

$$F\left(A - \hat{A}\right) = 0, \tag{38}$$

where $\left(A - \hat{A}\right)$ is a $U(1)$ connection on a trivial bundle.

Moreover if $\hat{A}_1$ and $\hat{A}_2 \in \mathcal{A}$ and are on the same $U(1)$ principle bundle, then

$$w\left(\hat{A}_1\right) \geq w\left(\hat{A}_2\right) \geq w\left(\hat{A}_1\right). \tag{39}$$

Consequently, $\hat{A}_2 - \hat{A}_1$ is a flat connection on a trivial bundle.

We will denote $\{A\}$ the equivalence class of connections on a given $U(1)$ principle bundle defined by elements which differ on a flat connection on a trivial bundle.

The functional (18) may be defined over the equivalence classes $\{A\}$, assuming we are only considering $U(1)$ connections on the same principle bundle over M.

If there is $\hat{A} \in \mathcal{A}$ on a given $U(1)$ principle bundle over M, $\{A\}$ is a strict local minimum of the Born-Infeld action (18).

In fact, we obtain after some calculations

$$S\left(\hat{A} + \delta A\right) - S\left(\hat{A}\right) \;=\; \hat{W}^{-\frac{3}{2}} \int_M \left[\hat{W} \sum_{m=1}^n a_m \delta P_m \wedge {}^*\delta P_m \right.$$

$$-\sum_{m=1}^{n} a_m \delta P_m \wedge {}^*\hat{P}_m \sum_{l=1}^{n} a_l{}^* \left(\delta P_l \wedge {}^*\hat{P}_l\right)\Bigg]$$

$$= \hat{W}^{-\frac{3}{2}} \int_M \sum_{m=1}^{n} a_m \delta P_m \wedge {}^*\delta P_m$$

$$+ \hat{W}^{-\frac{3}{2}} \int_M \Bigg[\sum_{m=1}^{n} a_m \hat{P}_m \wedge {}^*\hat{P}_m \sum_{l=1}^{n} a_l{}^* \left(\delta P_l \wedge {}^*\delta P_l\right)$$

$$- \sum_{m=1}^{n} a_m \delta P_m \wedge {}^*\hat{P}_m \sum_{l=1}^{n} a_l{}^* \left(\delta P_l \wedge {}^*\delta P_l\right) \Bigg]. \qquad (40)$$

The second integral term in the right hand member of (40) is the discriminant of the roots of the second order equation in λ

$$\int_M \left(P_m + \lambda \delta P_m\right) \wedge {}^* \left(P_m + \lambda \delta P_m\right) = 0. \qquad (41)$$

It is then greater or equal to zero.

Consequently we obtain

$$S\left(\hat{A} + \delta A\right) - S\left(\hat{A}\right) \geq \hat{W}^{-\frac{3}{2}} \int_M \sum_{m=1}^{n} a_m \delta P_m \wedge {}^*\delta P_m \geq 0. \qquad (42)$$

Moreover

$$S\left(\hat{A} + \delta A\right) = S\left(\hat{A}\right)$$

if and only if

$$F\left(A\right) = 0,$$

showing that $\{A\}$ is a strict local minimum of (18).

5   Minima of the Hamiltonian of Born-Infeld actions

In this section we will show that the previous construction of the minima of the Born-Infeld (B-I) actions over even D dimensional riemannian space may be performed for the

11

Hamiltonian of a B-I actions in $D+1$ pseudo-riemannian space-time. We will thus show that the generalized self-dual configurations are the minima of the Hamiltonian. We discuss this problem for a B-I action over $M_2 \times \mathbb{R}$ and $M_4 \times \mathbb{R}$ space-times where M_2 and M_4 are 2 and 4 dimensional compact riemannian spaces respectively. The Hamiltonian analysis of the Born-Infeld theory has been considered in several references [16, 17, 18].

We will consider the metrics in the ADM parametrization where

$$
\begin{aligned}
g_{00} &= -N^2 + \gamma_{ab} N^a N^b \\
g_{ab} &= \gamma_{ab} \\
g_{0a} &= g_{a0} = \gamma_{ab} N^b.
\end{aligned}
\tag{43}
$$

It is well know that in all p-brane formulations in the light cone gauge one obtains

$$
\begin{aligned}
N &= 1 \\
N^a &= 0.
\end{aligned}
$$

We will then work on that assumption which is in fact valid if we think that the B-I action will be used as a model for D-brane theories.

We start with the B-I action

$$
S = \frac{1}{b^2} \int_{M_2 \times \mathbb{R}} \left[\sqrt{-g} - \sqrt{-\det\left(g_{\alpha\beta} + bF_{\alpha\beta}\right)} \right] d^3x
\tag{44}
$$

We thus obtain for the conjugate momenta to (A_a), over $M_2 \times \mathbb{R}$,

$$
\pi^a = -\frac{\sqrt{\gamma} F^{0a}}{\left(1 + \frac{1}{2} b^2 F_{\alpha\beta} F^{\alpha\beta}\right)^{\frac{1}{2}}}
\tag{45}
$$

where the greek indices denote $\alpha = (0, a)$.

(45) may be solved for the time derivatives, it yields

$$\partial_0 A_a = \partial_a A_0 + \frac{\left(1 + \frac{1}{2}b^2 F_{cd} F^{cd}\right)^{\frac{1}{2}}}{\left(1 + b^2 \frac{1}{\gamma}\pi_c \pi^c\right)^{\frac{1}{2}}} \frac{\pi_a}{\sqrt{\gamma}}. \tag{46}$$

We finally obtain the Hamiltonian for the B-I action over $M_2 \times \mathbb{R}$

$$H = \frac{1}{b^2} \int_{M_2} \left\{ \left[1 + \frac{1}{2}b^2 F_{ab} F^{ab}\right]^{\frac{1}{2}} \left[1 + b^2 \frac{\pi_c \pi^c}{\gamma}\right]^{\frac{1}{2}} - 1 \right\} \sqrt{\gamma} d^2 x \tag{47}$$

subject to the first class constraint

$$\phi \equiv \partial_a \pi^a = 0. \tag{48}$$

We will now analyze the minima of (47), (48). They are static solutions to the canonical field equations.

Variations of (47) subject to (48) with respect to π_a yield the right hand side member of (46), A_0 being the Lagrange multiplier associated to (48). A solution to the stationary point condition is then

$$A_0 = \text{constant} \tag{49}$$

$$\pi^a = 0. \tag{50}$$

Variations with respect to A_a give, the stationary point condition

$$\partial_a \left[\frac{\left(1 + b^2 \frac{1}{\gamma}\pi_c \pi^c\right)^{\frac{1}{2}}}{\left(1 + \frac{1}{2}b^2 F_{ab} F^{ab}\right)^{\frac{1}{2}}} \sqrt{\gamma} F^{ab} \right] = 0. \tag{51}$$

If we now impose (50) into (51) we obtain the Lagrangean field equations for the euclidean B-I action over M_2 discussed in section 2. We then conclude that the generalized self-dual configurations are stationary points of the hamiltonian of the B-I action.

We will now consider the second variation of (47) subject to (51) around those solutions to prove that these configurations are indeed local minima of the hamiltonian. The only contribution to the second variation of (47) containing variations of π^a is of the form $\delta\pi_a\delta\pi^a$, since $\pi^a = 0$. Its coefficient is manifestly positive definite. The other contributions comes from the second variations of

$$\frac{1}{b^2}\int_{M_2}\left\{\left[1+\frac{1}{2}b^2F_{ab}F^{ab}\right]^{\frac{1}{2}}-1\right\}\sqrt{\gamma}d^2x, \tag{52}$$

but this is exactly the B-I action over euclidean space considered in section 2 and 3, where it was show that these configurations are minima of it. We thus conclude that these configurations are minima of the Hamiltonian of the B-I action over $M_2\times\mathbb{R}$.

We may now analyze the problem over $M_4\times\mathbb{R}$. The conjugate momenta to A_a is given by

$$\pi^a = \frac{-\sqrt{\gamma}\left(F^{0a}+b^2F^{0c*}F_{cb}{}^*F^{ab}\right)}{\left[1+\frac{1}{2}b^2F_{\alpha\beta}F^{\alpha\beta}+\frac{1}{16}b^4F^{\mu\nu*}F_{\mu\nu\gamma}{}^*F^{\gamma\sigma\lambda}F_{\sigma\lambda}\right]^{\frac{1}{2}}}. \tag{53}$$

We proceed now to construct the Hamiltonian. It becomes difficult to invert the time derivative of A_a in terms of π^a. To avoid this step we consider the following density

$$\mathcal{H} = \frac{1}{b^2}\left[\sqrt{-\det\left(g_{\alpha\beta}+bL_{\alpha\beta}\right)}-\sqrt{-\det g_{\alpha\beta}}\right] \tag{54}$$

where $L_{\alpha\beta}$ is an antisymmetric tensor, with

$$L_{0a} = \mathrm{i}\frac{\pi_a}{\sqrt{\gamma}}$$
$$L_{ab} = {}^*F_{ab}. \tag{55}$$

$\mathcal{H}$ will be the correct Hamiltonian density if after replacing of (53) into the canonical action

$$S_c = \int_{M_4\times\mathbb{R}}\left[F_{0a}\pi^a-\mathcal{H}\right]d^5x \tag{56}$$

14

we recover the B-I action.

We will show that this is the case. To do so, consider the relations

$$-\det\left(g_{\alpha\beta}+bL_{\alpha\beta}\right) = \det\left(\gamma_{ab}+bF_{ab}\right)+b^2\left(\pi_a\pi^a+b^2\pi_b\pi^cF^{bd}F_{cd}\right) \tag{57}$$

$$-\det\left(g_{\alpha\beta}+bF_{\alpha\beta}\right) = \det\left(\gamma_{ab}+bF_{ab}\right)+\gamma b^2\left(F_{0a}F^{0a}+b^2F_{0a}{}^*F^{ab*}F_{cb}F^{0c}\right). \tag{58}$$

After several calculations we obtain

$$\left[-\det\left(g_{\alpha\beta}+bF_{\alpha\beta}\right)\right]\left[-\det\left(g_{\alpha\beta}+bL_{\alpha\beta}\right)\right]=\left[\det\left(\gamma_{ab}+bF_{ab}\right)\right]^2. \tag{59}$$

We then use this formula to replace $\mathcal{H}$ in (56), we also replace π^a by (53) we then exactly obtain the B-I action (44) over $M_4\times\mathbb{R}$.

We will now show that the extended self-dual fields are minimal configurations of the Hamiltonian

$$H=\int_{M_4}\mathcal{H}d^4x. \tag{60}$$

The stationary point conditions for H yield

$$\partial_a A_0 = \frac{\pi_a+b^2\pi_bF^{cb}F_{ac}}{\sqrt{-\det\left(g_{\alpha\beta}+bL_{\alpha\beta}\right)}}, \tag{61}$$

from which we consider the solutions

$$A_0 = \text{constant}$$

$$\pi^a = 0,$$

and

$$\partial_a\left\{\frac{\gamma\left[F^{ab}+\frac{1}{4}b^2F^{cd*}F_{cd}{}^*F^{ab}-b^2\frac{1}{\gamma}\pi_c\pi^bF^{ca}+b^2\frac{1}{\gamma}\pi_c\pi^aF^{cb}\right]}{\sqrt{-\det\left(g_{\alpha\beta}+bL_{\alpha\beta}\right)}}\right\}=0 \tag{62}$$

which reduces, after elimination of π^a, to the B-I fields equations over M_4 considered in section 2. Consequently the extended self-dual configurations are solutions to the stationary point condition for H.

We now consider the second variation of H. The terms containing $\delta\pi^a$ necessarily are of the form

$$\left(\frac{1}{\gamma}\delta\pi_a\delta\pi^a + b^2\frac{1}{\gamma}\delta\pi_b\delta\pi^c F^{bd}F_{cd}\right)$$

with a positive coefficient. They are then strictly positive. The others contributions come from the second variation of

$$\frac{1}{b^2}\int\limits_{M_4}\left[\sqrt{\det\left(\gamma_{ab}+bF_{ab}\right)} - \sqrt{\det\gamma_{ab}}\right]d^4x$$

which is exactly the B-I action over euclidean space analyzed in section 2 and 3. We then conclude that the extended self-dual configurations over M_4 are local strict minima of the hamiltonian (54) provided that 1-form connections over the same principle bundle are considered.

6 Conclusions

We constructed a class of stable exact solutions to the non-linear Born-Infeld field equations. They leave invariant the set of polinomic terms in the curvature

$$P_m = F \wedge \ldots \wedge F, \qquad m = 0, \ldots, n \tag{63}$$

under the Hodge dual operation. In this sense they are an extension to other dimensions of the self-duality configurations. We proved they are local minima of the Born-Infeld action for riemannian manifolds and minima of the Hamiltonian for pseudo-riemannian ones. The construction is obtained by explicit use of a general expression for the Born-Infeld determinant derived in section 2.

The extended self-dual configurations are defined for even dimensions only. As in the case of self-dual ones they may or may not exist on a given manifold with a given metric. They exist for any kählerian manifold. We will discuss elsewhere related configurations which minimize the Born-Infeld action over odd dimensional space-times. We expect the extended self-dual configurations will correspond to minima of related D-brane actions. They certainly do correspond for the $2D$-brane in $D = 9$ as was shown in [13].

The paper [19] appeared after we had finished our work. The $D = 4$ case in that paper overlaps with our analysis, which is valid for any even dimension.

References

[1] Born, M. & Infeld, L. *Proc. R. Soc. London*, **A144** (1934) 425.

[2] Gibbons, G. W. & Rasheed, D. *Nucl. Phys.*, **B454** (1995) 185.

[3] Gaillard, M. K. & Zumino, B. *"Self-Duality in Nonlinear Electromagnetism"*, hep-th/9705226.

[4] Brace, D., Morariu, B. & Zumino, B. *"Duality Incariant Born-Infeld Theory"*, hep-th/9905218.

[5] Tseytlin, A. *Nucl. Phys.*, **B501** (1997) 47.

[6] Dai, J., Leigh, R. G., & Polchinski, J. *Mod. Phys. Lett.*, **A4** (1989) 2073.

[7] Leigh, R. G. *Mod. Phys. Lett.*, **A4** (1989) 675.

[8] Schmidhuber, C. *"D-brane actions"*, hep-th/9601003.

[9] Polchinski, J. *Phys. Rev. Lett.*, **75** (1995) 4724.

[10] Polchinski, J. *"Tasi Lectures on D-branes"*, hep-th/9611050.

[11] Trautmant, A. *Int. J. Theo. Phys.*, **16** (1977) 561.

[12] Martín, I. & Restuccia, A. *Lett. Math. Phys.*, **39** (1997) 379.

[13] Martín, I., Ovalle, J. & Restuccia, A. *Phys. Lett. B*, **472** (2000) 77.

[14] Martín, I., Restuccia, A. & Torrealba, R. *Nucl. Phys. B* **521** (1998) 117.

[15] Bellorín, J. A. *"Solutions of Born-Infeld equations over $\mathbb{C}P_n$"*. Preprint Universidad de Oriente, Departamento de Física. Venezuela. (2000).

[16] Bengtsson, I. *Int. J. Mod. Phys.*, **A12** (1997) 4869.

[17] Khoudeir, A. & Parra, Y. *Phys. Rev. D*, **58** (1998) 025010.

[18] De Castro, A. & Restuccia, A. *Class. Quant. Grav.*, **16** (1999) 2729.

[19] Gibbons, G. W. & Hashimoto, K. *"Non-Linear Electrodynamics in Curved Backgrounds"*. hep-th/0007019.